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Embedding surfaces, handlebodies and finite graphs into S^3 with maximum symmetry

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Abstract

We restrict our discussion to the orientable category. For $g > 1$, let OE_g be the maximum order of a finite group G acting on the closed surface Σ_g of genus g which extends over (S^3, Σ_g), for all possible embeddings $\Sigma_g \hookrightarrow S^3$. Then OE_g is given in the table below.

OE_g	g
$12(g-1)$	$2,3,4,5,6,9,11,17,25,97,121,241,601$
$8(g-1)$	$7,49,73$
$20(g-1)/3$	$16,19,361$
$6(g-1)$	$21,481$
192	41
7200	1681
$4(\sqrt{g}+1)^2$	$g=k^2, k \neq 3,5,7,11,19$
$4(g+1)$	the remaining numbers

Moreover OE_g can be realized by unknotted embeddings for all g except for $g = 21$ and 481.

The same list holds for embeddings of 3-dimensional handlebodies of genus $g > 1$ into S^3, and also for embeddings of finite graphs into S^3 where $g > 1$ denotes now the rank of the free fundamental group of the graph.

Contents

1 Introduction

Surfaces belong to the most familiar topological subjects to us, mostly because we can see them staying in our 3-space in various manners. The symmetries of the surfaces have been studied for a long time, and it will be natural to wonder when these symmetries can be embedded into the symmetries of our 3-space (3-sphere). In particular, what are the orders of the maximum symmetries of surfaces which can be embedded into the symmetries of the 3-sphere S^3? We will solve this maximum order problem in this paper in the orientable category.

We use Σ_g (V_g) to denote the closed orientable surface (handlebody) of genus $g > 1$, and G to denote a finite group acting on Σ_g or on an orientable 3-manifold. The actions we consider are always faithful and orientation-preserving on both surfaces and 3-manifolds. We are always working in the smooth category. By the geometrization of finite group actions in dimension 3, for actions on the 3-sphere, we can then restrict to orthogonal actions.

Let O_g be the maximal order of all finite groups which can act on Σ_g. A classical result of Hurwitz states that O_g is at most $84(g-1)$ [Hu]. However, to determine O_g is still a hard and famous question in general, and there are numerous interesting partial results.

Let OH_g be the maximal order of all finite groups which can act on V_g. It is a result due to Zimmermann [Zi] that $4(g+1) \le OH_g \le 12(g-1)$, see also [MMZ], moreover OH_g is either $12(g-1)$ or $8(g-1)$ if g is odd, and each of them is achieved by infinitely many odd g [MZ]. However, in general OH_g are still not determined either.

In [WWZZ], we considered finite group actions on the pair (S^3, Σ_g), with respect to an embedding $e : \Sigma_g \hookrightarrow S^3$. If G can act on the pair (S^3, Σ_g) such that its restriction on Σ_g is the given G-action on Σ_g, we call the action of G on Σ_g extendable (over S^3 with respect to e).

Call an embedding $e : \Sigma_g \hookrightarrow S^3$ standard (or unknotted), if each component of $S^3 \setminus e(\Sigma_g)$ is a handlebody, otherwise it is knotted. Similarly, we define an action of G on V_g to be extendable and the embedding $e : V_g \hookrightarrow S^3$ to be standard or knotted. For each g, the standard embedding is unique up to isotopy of S^3 and automorphisms of Σ_g (resp. V_g).

Let OE_g be the maximal order of all extendable finite groups acting on Σ_g. Let OE_g^s be the maximal order of all finite group actions on Σ_g which extend over S^3 w.r.t. the standard embedding. Then we know that $4(g+1) \le OE_g^s \le OH_g \le 12(g-1)$, and there are only finitely many g such that $OE_g^s = 12(g-1)$; moreover, $OE_g^s \ge 4(n+1)^2$ for each $g = n^2$ [WWZZ].

In this paper we will determine OE_g for all $g > 1$ (Theorem 6.3). Actually we use a primary observation that, if G acts on (S^3, Σ_g) with $|G| > 4(g-1)$, then Σ_g bounds a handlebody (Proposition 2.5), and we will give a classification of group actions on (S^3, Σ_g) with $|G| > 4(g-1)$ in a

certain sense (Theorem 6.1). And by such a classification we can determine OE_g^s and OE_g^k (Theorem 6.4 and Theorem 6.5), where OE_g^k denotes the maximal order of finite group actions on Σ_g which extend over S^3 w.r.t. all possible knotted embeddings.

Then some interesting phenomena appear: As expected, for all g with finitely many exceptions we have $OE_g^s > OE_g^k$; indeed there are only finitely many g such $OE_g^s = OE_g^k$ and, a little bit surprising, $OE_g^s < OE_g^k$ when $g = 21$ or 481. Also for some g, $OE_g^k = 12(g-1)$.

Our approach relies on the orbifold theory which is founded and studied in [Th], [Du1], [Du2], [BMP] and [MMZ]. More precisely, the proof of our main results translates into the problem of finding the so-called *allowable* 2-orbifolds (Definition 4.1) in certain spherical 3-orbifolds. The strategy of such an approach will be given in Section 4.

In Section 2, after introducing some basic notions about orbifolds and finite group actions on manifolds, we present a sequences of observations concerning the orbifold pair $(S^3, \Sigma_g)/G$ on both the topological level and the group theoretical level which are very useful for our later approach. In Section 3 we will describe Dunbar's list of spherical 3-orbifolds whose underlying space is S^3. With the material prepared in Sections 2 and 3, we will be able to explain why we can transfer the problem of finding OE_g into the problem of finding allowable 2-orbifolds in certain spherical 3-orbifolds and, more importantly, to outline how to get a practical method to find such 2-orbifolds. In Section 5 we will give the list of 3-orbifolds containing allowable 2-suborbifolds which turns out to be a small subset of Dunbar's list where the singular sets are relatively simple. In Section 6, we will find all allowable 2-orbifolds in the list of 3-orbifolds provided by Section 5, and then the main results are derived. We end the paper by some knotted examples.

We emphasize that the main results of the present paper, Theorems 6.2-6.5, hold also for embeddings of 3-dimensional handlebodies of genus $g > 1$ into S^3; in fact, all surfaces embedded into S^3 considered in the present paper turn out to bound a handlebody (just on one side in the knotted case, on both sides in the unknotted one). Moreover the main results, Theorems 6.2-6.5, hold also for embeddings of finite graphs into S^3 where $g > 1$ now denotes the rank of the free fundamental group of the graph. In fact, a regular neighborhood of an embedded graph in S^3 of rank g is a handlebody of genus g, and viceversa all handlebodies considered in the present paper are constructed as regular neighborhoods of finite graphs.

So it is easy to go from finite embedded graphs to handlebodies and then to surfaces; by the methods of the present paper one can go also in the other direction, that is from embedded surfaces to handlebodies and then to finite graphs.

We will state our main results only for surfaces in the present paper, the other two cases being completely analogous. We are planning a paper which classifies the rank g abstract hyperbolic graphs Γ (that is, of rank $g > 1$ and without free edges), and their embeddings $\Gamma \to S^3$ which realize OE_g, from point of view of graph theorists.

Remark 1.1. (1) The maximum order of a finite group of automorphisms of

a hyperbolic graph is $2^g\, g!$, $g > 2$ [WZ], realized by the automorphism group of the $(g+1)$-petaled rose.

(2) Restricting the above problems to cyclic and abelian group actions, the answers are known and simpler, see [WWZZ] for details.

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2 Orbifolds and finite group actions

The orbifolds we consider have the form M/H, where M is a n-manifold and H is a finite group acting faithfully on M. For each point $x \in M$, denote its stable subgroup by $St(x)$, its image in M/H by x'. If $|St(x)| > 1$, x' is called a singular point and the singular index is $|St(x)|$, otherwise it is called a regular point. If we forget the singular set we get a topological space $|M/H|$ which is called underlying space. M/H is orientable if M is orientable and H preserve the orientation; M/H is connected if $|M/H|$ is connected.

We can also define covering spaces and fundamental group for an orbifolds. There is a one to one correspondence between orbifold covering spaces and conjugacy classes of subgroup of the orbifold fundamental group, and regular covering spaces correspond to normal subgroups. A Van-Kampen theorem is also valid, see [BMP]. In the following, covering space or fundamental group refers always to the orbifold setting.

Definition 2.1. A discal orbifold (spherical orbifold) has the form B^n/H (S^n/H), where B^n (S^n) is the n-dimension ball (sphere) and H a finite group acting orientation-preservingly on the corresponding manifold. A handlebody orbifold has the form V_g/H.

By a classical result for topological actions, $|B^2/H|$ is a disk, possibly with one singular point. Since $SO(3)$ contains only five classes of finite subgroups: the order n cyclic group C_n, the order $2n$ dihedral group D_n, the order 12 tetrahedral group T, the order 24 octahedral group O, and the order 60 icosahedral group J, it is easy to see B^3/H (S^2/H) belongs to one of the following five models. The underlying space $|B^3/H|$ ($|S^2/H|$) is the 3- ball (the 2-sphere).

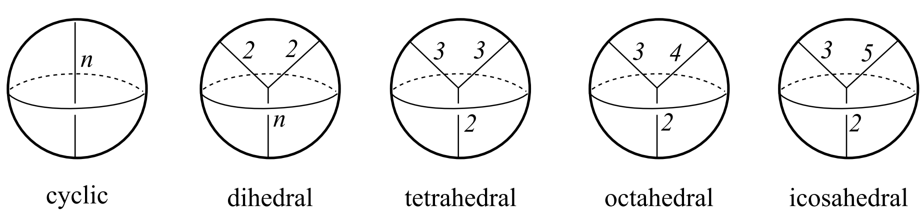


Figure 1

By the equivariant Dehn lemma, see [MY], it is easy to see that a handlebody orbifold is the result of gluing finitely many 3-discal orbifolds along some 2-discal orbifolds. And such gluing respecting orientations always gives us a handlebody orbifold.

Like in the manifold case we can say that an orientable seperating 2-suborbifold $\mathcal{F}$ in an orientable 3-orbifold $\mathcal{O}$ is standard or knotted, depending on whether it bounds handlebody orbifolds on both sides.

It is easy to see that if the underlying space of a handlebody orbifold is a ball, then the singular set would form an unknotted tree in the ball, possibly disconnected. Unknotted means the complement of the regular neighborhood of the singular set is a handlebody. For more about handlebody orbifold theory one can see [MMZ].

Suppose the action of G on Σ_g is extendable w.r.t. some embedding $e : \Sigma_g \hookrightarrow S^3$; let $\tilde{\Gamma} = \{x \in S^3 \mid \exists g \in G, s.t.\, gx = x\}$. As locally there are only five kinds of model, $\tilde{\Gamma}$ is a graph, possibly disconnected, and S^3/G is a 3-orbifold whose singular set $\Gamma = \tilde{\Gamma}/G$ a trivalent graph. Each edge of Γ can be labeled by an integer $n > 1$ which indicates its singular index. At each vertex the labels m, q, r of the three adjacent edges should satisfy $1/m + 1/q + 1/r > 1$. The 2-orbifold Σ_g/G maps to the 2-suborbifold $e(\Sigma_g)/G$ whose singular set $e(\Sigma_g)/G \cap \Gamma$ consists of isolated points.

We then have an orbifold covering $p : S^3 \to S^3/G$ and an orbifold embedding $e/G : \Sigma_g/G \hookrightarrow S^3/G$. Conversely, if we have an orbifold embedding from a 2-orbifold to a spherical orbifold and the preimage of the 2-suborbifold in S^3 is connected then we find an extendable action of G on some surface with respect to some embedding.

Definition 2.2. An orientable 2-suborbifold $\mathcal{F}$ in an orientable 3-orbifold $\mathcal{O}$ is compressible if either $\mathcal{F}$ is spherical and bounds a discal 3-suborbifold in $\mathcal{O}$, or there is a simple closed curve in $\mathcal{F}$ (not meeting the singular set) which bounds a 2-discal orbifold in $\mathcal{O}$, but does not bound a 2-discal orbifold in $\mathcal{F}$. Otherwise $\mathcal{F}$ is called incompressible.

Lemma 2.3. *Any 2-suborbifold $\mathcal{F}$ in a spherical orbifold S^3/G is compressible.*

Proof. $\mathcal{F}$ cuts S^3/G into two parts $\mathcal{O}_1$, $\mathcal{O}_2$, and $p^{-1}(\mathcal{F})$ cuts S^3 into several components $M_1, M_2, \cdots, M_k$, each of which will be mapped by p to one of the two parts.

If $\mathcal{F}$ is spherical, $p^{-1}(\mathcal{F})$ is a disjoint union of 2-spheres. By the irreducibility of S^3 and B^3, one M_i must be a ball, hence one $\mathcal{O}_i$ is a discal 3-suborbifold and we have the result by definition.

Otherwise, $F = p^{-1}(\mathcal{F})$ is a disjoint union of homeomorphic closed surfaces in S^3 of genus $g \geq 1$. Since F is compressible in S^3 we can find an innermost compressing disk D. Suppose D is in M_i. By the equivariant Dehn Lemma we can find equivariant compressing disks in M_i. The image of these disks in S^3/G is a 'compressing disk' in the orbifold. □

Lemma 2.4. *Suppose $\mathcal{F}$ is a 2-suborbifold of a spherical orbifold S^3/G and $|\mathcal{F}|$ is homeomorphic to S^2.*

(1) If $\mathcal{F}$ has not more than three singular points then $\mathcal{F}$ is spherical and bounds a discal 3-orbifold.

(2) If $\mathcal{F}$ has four singular points then $\mathcal{F}$ bounds a handlebody orbifold in S^3/G.

Proof. As a 2-suborbifold, $\mathcal{F}$ should be spherical or has 'compressing disk' by Lemma 2.3.

(1) If $\mathcal{F}$ has no more than three singular points, every simple closed curve in $\mathcal{F}$ bounds a discal orbifold in $\mathcal{F}$. So $\mathcal{F}$ has no 'compressing disk' and hence is spherical, and then bounds a discal 3-orbifold.

(2) If $\mathcal{F}$ has four singular points, $\mathcal{F}$ is not spherical and hence has a 'compressing disk' D. Then ∂D separate $\mathcal{F}$ into two discal orbifolds D_1, D_2, each of which contains two singular points. Now $D_1 \cup D$ and $D_2 \cup D$ are 2-suborbifolds in S^3/G each of which contains no more than three singular points; by the above argument each of them bounds a discal 3-orbifold. There are two cases: in the first case the two discal 3-orbifolds meet only along the boundary, and the result is clearly a handlebody orbifold; in the other case one is contained in the other, and again we get a handlebody orbifold. In the later case the singular set of the handlebody orbifold contains two arcs. Hence, $\mathcal{F}$ bounds a handlebody orbifold in S^3/G. □

Proposition 2.5. *Suppose G acts on (S^3, Σ_g). If $|G| > 4(g-1)$, then Σ_g/G has underlying space S^2 with four singular points and bounds a handlebody orbifold, and Σ_g bounds a handlebody.*

In conclusion $OE_g \leq OH_g \leq 12(g-1)$.

Proof. Σ_g/G is a 2-suborbifold in S^3/G whose singular set contains isolated points $a_1, a_2, \cdots, a_k$, with indices $q_1 \leq q_2 \leq \cdots \leq q_k$. Note that $|S^3/G|$ and $|\Sigma_g/G|$ are both manifolds. Suppose the genus of $|\Sigma_g/G|$ is $\hat{g}$. By the Riemann-Hurwitz formula

$$2 - 2g = |G|(2 - 2\hat{g} - \sum_{i=1}^{k}(1 - \frac{1}{q_i}))$$

we have

$$|G| = (2g-2)/(\sum_{i=1}^{k}(1 - \frac{1}{q_i}) + 2\hat{g} - 2).$$

If $\hat{g} \geq 1$ or $\hat{g} = 0$, $k \geq 5$, then $|G| \leq 4g - 4$. Hence $\hat{g} = 0$ and $k \leq 4$. If $k \leq 3$ then Σ_g/G bounds a discal orbifold by Lemma 2.4 (1), which leads to a contradiction (since $g > 1$ by assumption). Hence $k = 4$, and by Lemma 2.4 (2) Σ_g/G bounds a handlebody orbifold. In this case Σ_g bounds a handlebody in S^3.

By [WWZZ], $OE_g \geq 4(g+1)(> 4(g-1))$. Hence each Σ_g in S^3 realizing OE_g must bound a handlebody, and therefore $OE_g \leq OH_g \leq 12(g-1)$. □

Definition 2.6. Let $\mathcal{F}$ be a 2-suborbifold in a spherical orbifold S^3/G, with $|\mathcal{F}|$ homeomorphic to S^2 and four singular points a_1, a_2, a_3, a_4. Supposing $q_1 \leq q_2 \leq q_3 \leq q_4$ for their indices, we call (q_1, q_2, q_3, q_4) the singular type of $\mathcal{F}$.

Using the Riemann-Hurwitz formula, it is easy to see:

Lemma 2.7. *If $|G| > 4(g-1)$ then the singular type of Σ_g/G is one of* $(2,2,2,n)(n \geq 3)$, $(2,2,3,3)$, $(2,2,3,4)$, $(2,2,3,5)$.

Lemma 2.8. *The relation between the orders of extendable group actions and the surface genus for a given singular type is given in the following table:*

Type	$(2,2,2,n)(n \geq 3)$	$(2,2,3,3)$	$(2,2,3,4)$	$(2,2,3,5)$
Order	$4n(g-1)/(n-2)$	$6(g-1)$	$24(g-1)/5$	$30(g-1)/7$

Lemma 2.9. *If the singular type of Σ_g/G is not $(2,2,3,3)$, the handlebody orbifold bounded by Σ_g/G is as in Figure 2(a); if the singular type is $(2,2,3,3)$, there are the two possibilities in Figure 2(a) and (b) for this handlebody orbifold.*

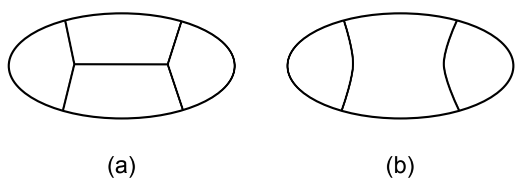


Figure 2

Note that in the case of Figure 2(a) the handlebody orbifold is a regular neighborhood of a singular edge.

Lemma 2.10. *Suppose a finite group G acts on (M, F), where M is a 3-manifold, with a surface embedding $i : F \hookrightarrow M$, so we have diagrams:*

$$\begin{tikzcd} F \arrow[r, "i"] \arrow[d, "p"'] & M \arrow[d, "p"] \\ F/G \arrow[r, "\hat{\imath}"] & M/G \end{tikzcd} \qquad \begin{tikzcd} \pi_1(F) \arrow[r, "i_*"] \arrow[d, "p_*"'] & \pi_1(M) \arrow[d, "p_*"] \\ \pi_1(F/G) \arrow[r, "\hat{\imath}_*"] & \pi_1(M/G) \end{tikzcd}$$

Suppose F/G is connected. Then F is connected if and only if

$$\hat{\imath}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M)) = \pi_1(M/G).$$

Proof. " $\Longrightarrow$ " Suppose $\hat{\imath}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M)) \subsetneqq \pi_1(M/G)$. We can find an orbifold covering space $\widehat{M}$ corresponds to $\hat{\imath}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M))$. Then we have diagram:

$$\begin{tikzcd} F \arrow[rr, "i"] \arrow[dd, "p"'] & & M \arrow[dl] \arrow[dd, "p"] \\ & \widehat{M} \arrow[dr, "\hat{p}"] & \\ F/G \arrow[ur, dashed] \arrow[rr, "\hat{\imath}"] & & M/G \end{tikzcd}$$

Since $\hat{\imath}_*(\pi_1(F/G)) \subseteq \hat{p}_*(\pi_1(\widehat{M}))$, F/G can lift to $\widehat{M}$, and it lifts to a disjoint union of copies. Hence F must be disconnected.

" $\Longleftarrow$ " Suppose F is not connected. Let $F_1 \subseteq F$ be a component of F and G_1 its stablilizer in G, that is $G_1 = \{h \in G \mid h(F_1) = F_1\}$. Then $F_1/G_1 = F/G$. Now $|\pi_1(M/G) : p_*(\pi_1(M))| = |G|$, and

$$\begin{aligned}
& |\hat{i}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M)) : p_*(\pi_1(M))| \\
= & |\hat{i}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M))/p_*(\pi_1(M))| \\
= & |\hat{i}_*(\pi_1(F/G))/\hat{i}_*(\pi_1(F/G)) \cap p_*(\pi_1(M))| \\
\leq & |\hat{i}_*(\pi_1(F/G)) : \hat{i}_* p_*(\pi_1(F_1))| \\
= & |\pi_1(F/G)/ker\hat{i}_* : p_*(\pi_1(F_1)) \cdot ker\hat{i}_*/ker\hat{i}_*| \\
= & |\pi_1(F_1/G_1) : p_*(\pi_1(F_1)) \cdot ker\hat{i}_*| \\
\leq & |\pi_1(F_1/G_1) : p_*(\pi_1(F_1))| \\
= & |G_1| < |G|.
\end{aligned}$$

Hence $\hat{i}_*(\pi_1(F/G)) \cdot p_*(\pi_1(M)) \subsetneqq \pi_1(M/G)$. ∎

Remark 2.11. (1) When F is connected, we have $\hat{i}_* p_*(\pi_1(F)) = \hat{i}_*(\pi_1(F/G)) \cap p_*(\pi_1(M))$ and $ker(\hat{i}_*) \subseteq p_*(\pi_1(F))$. If M is simply connected, then F is connected if and only if $\hat{i}_*$ is surjective.

(2) If $F/G \subset S^3/G$ bounds handlebody orbifolds on both sides then clearly $\hat{i}_*$ is surjective.

Corollary 2.12. *Suppose $\mathcal{F}$ is a connected 2-suborbifold with an embedding $\hat{i} : \mathcal{F} \hookrightarrow S^3/G$ into a spherical orbifold S^3/G. Let $p^{-1}(\mathcal{F}) = \Sigma$; then Σ is connected if and only if $\hat{i}_*$ is surjective.*

Lemma 2.13 (Edge killing)**.** *Let (X, Γ) be an orientable 3-orbifold with underlying space X and singular set a trivalent graph Γ. An edge killing operation is defined as in Figure 3, where (a, b) denotes the greatest common divisor of a and b. If via an edge killing operation we get from Γ a new graph Γ', then we have a surjective homomorphism $\pi_1(X, \Gamma) \to \pi_1(X, \Gamma')$.*

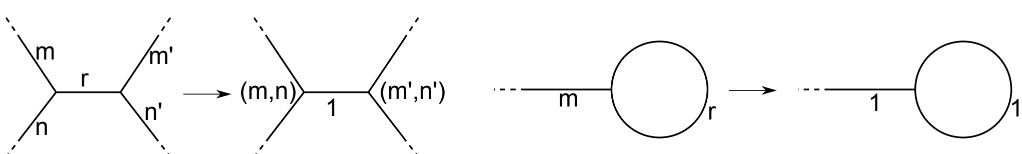


Figure 3

Proof. Denoting by $N(\Gamma)$ a regular neighborhood of Γ in X, there is a surjective homomorphism from $\pi_1(X - N(\Gamma))$ to $\pi_1(X, \Gamma)$, and we can compute $\pi_1(X, \Gamma)$ from $\pi(X - N(\Gamma))$ by adding relations like $x^r = 1$ [BMP]. The effect of an edge killing operation on fundamental groups is just adding relations like $x = 1$, and then we obtain a presentation of $\pi(X, \Gamma')$. ∎

Lemma 2.14. *Let G be an extendable finite group action with respect to some embedding $e : \Sigma_g \hookrightarrow S^3$. If $|e(\Sigma_g)/G|$ is homeomorphic to S^2, then $|S^3/G|$ is homeomorphic to S^3.*

Proof. By Corollary 2.12 the homomorphism $(e/G)_* : \pi_1(\Sigma_g/G) \to \pi_1(S^3/G)$ is surjective. By Lemma 2.13, if we kill all the singular edges we get a surjection $\pi_1(|\Sigma_g/G|) \to \pi_1(|S^3/G|)$. Hence $\pi_1(|S^3/G|)$ is trivial and $|S^3/G|$ is homeomorphic to S^3.

□

The following lemma will be used to prove that some group homomorphisms are not surjective.

Lemma 2.15. *Let S be one of the permutation groups A_4, S_4, A_5. Let H be a subgroup of $S \times S$ such that the restrictions to H of the two canonical projections of $S \times S$ to S are both surjective. If H is not isomorphic to S then an order 2 element and an order 3 element in H cannot generate H.*

Proof. Let (x, x') and (y, y') be order 2 and order 3 elements in H which generate H. Since the two projections restricted to H are surjective, both x and x' have order 2, and both y and y' have order 3; moreover the subgroups generated by x, y and also by x', y' are both equal to S. One can check now by explicit computations in each of the three groups that the map $x \mapsto x'$, $y \mapsto y'$ gives an isomorphism of S to itself. Hence H is isomorphic to S. □

Notice that $T \cong A_4$, $O \cong S_4$, $J \cong A_5$. We will use this lemma to some finite groups, with form $S \times S$, in $SO(3) \times SO(3)$ which is 2-sheet covered by $SO(4)$.

3 Dunbar's list of spherical 3-orbifolds

In [Du1], [Du2] Dunbar produces lists all spherical orbifolds with underlying space S^3. We list these pictures below and give a brief explanation such that one can check graphs conveniently. For more information, one should see the original paper.

Since the underlying space is S^3, all the information is contained in the trivalent graphs of the singular sets. Each edge in a graph is labeled by an integer indicating the singular index of the edge, with the convention that each unlabeled edge has index 2. If a graph has a vertex such that the incident edges have labels $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$, the orbifold is non-fibred. All the non-fibred spherical orbifolds have underlying space S^3 and are listed in Table 8. Otherwise the orbifolds are Seifert fibred and listed in Table 6 (the basis of the fibration is a 2-sphere) and Table 7 (the basis is a disk).

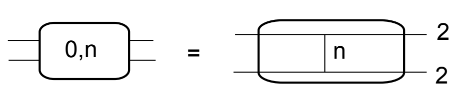
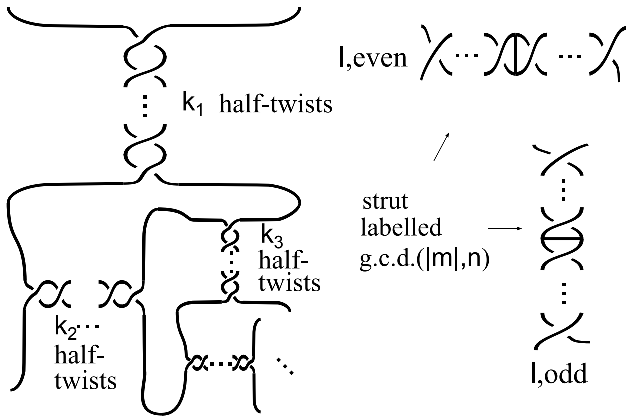


Figure 4

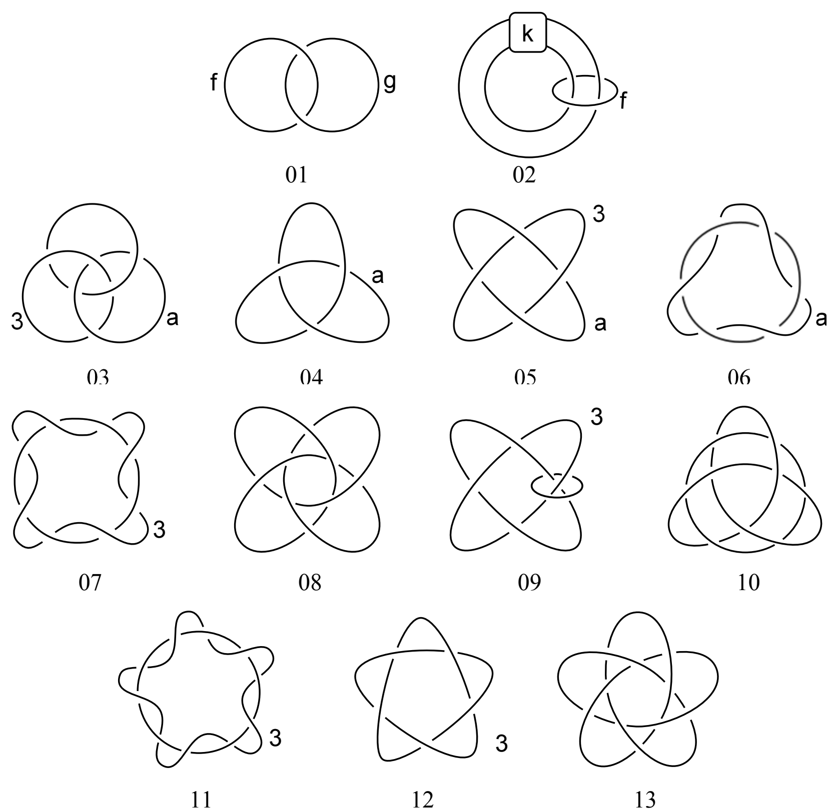


$$\frac{|m|}{n} = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{\ddots + \cfrac{1}{k_l}}}}$$

Figure 5

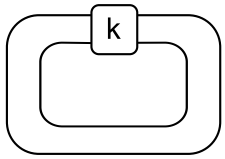
Now we give the lists of all spherical 3-orbifolds with underlying space S^3.

Table 6: Fibred case with base S^2



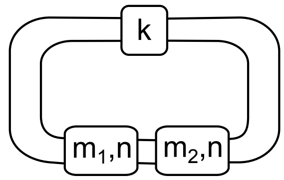
In Table 6 and Table 7 many graphs have some free or undetermined parameters (just called parameters in the following). These parameters should satisfy $n > 1$, $3 \le a \le 5$, $f \ge 1$, $g \ge 1$, and in Table 6 we require $k \ne 0$. The letter '@' means amphicheiral (there exists an orientation-reversing homeomorphism of the orbifold). If an orbifold is non-amphicheiral, as in the original paper its mirror image is not presented.

Table 7: Fibred case with base D^2



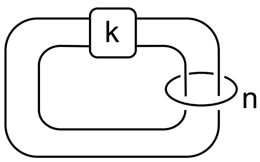
$k \neq 0$

14



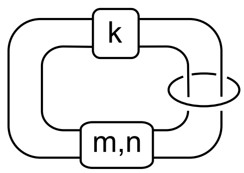
$k+m_1/n+m_2/n \neq 0$

15



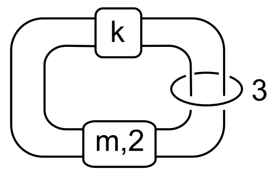
$k \neq 0$

16



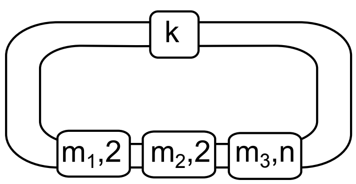
$k+m/n \neq 0$

17



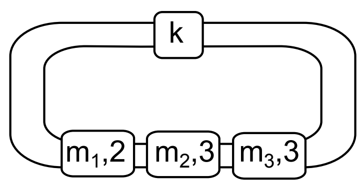
$k+m/2 \neq 0$

18



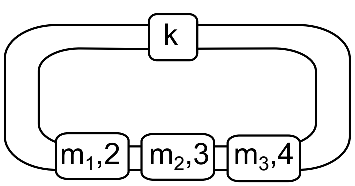
$k+m_1/2+m_2/2+m_3/n \neq 0$

19



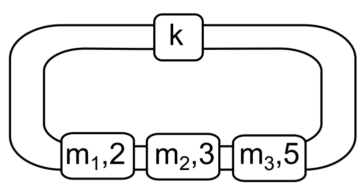
$k+m_1/2+m_2/3+m_3/3 \neq 0$

20



$k+m_1/2+m_2/3+m_3/4 \neq 0$

21

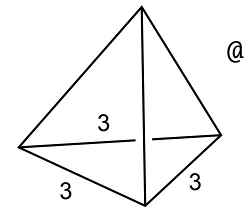


$k+m_1/2+m_2/3+m_3/5 \neq 0$

22

A box with an integer k indicates two parallel arcs with k-half twists, the over-crossings from lower left to upper right if $k > 0$, and upper left to lower right if $k < 0$. A box with two integer m, n stands for a picture as in Figure 4 and 5; It satisfies $|2m| \leq n$. All the crossing numbers of the horizontal and vertical parts are determined by the unique continued fraction presentation of $|m|/n$, such that all k_i are positive and $k_l \geq 2$. All the over-crossings are from lower left to upper right if $m > 0$, and from upper left to lower right if $m < 0$. If the greatest common divisor $(|m|, n) = d > 1$, we add a 'strut' labeled d in the k_l twist as shown in the picture. If $m = 0$, we add a 'strut' labeled n between two parallel lines.

Table 8: Non-fibred case



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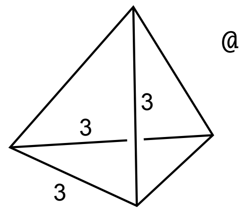

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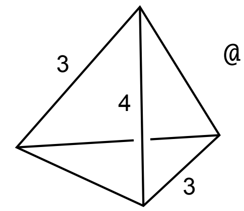

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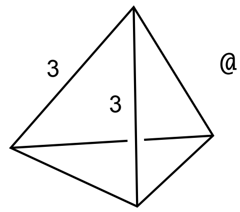

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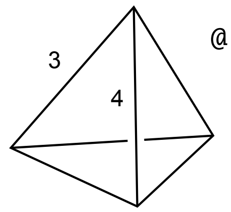

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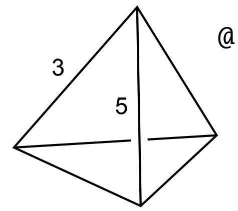

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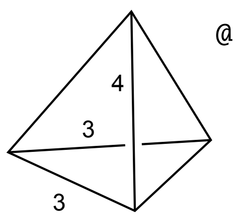

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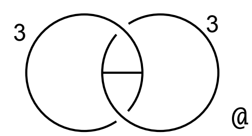

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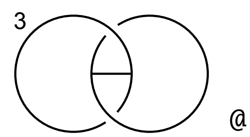

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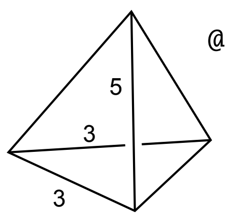

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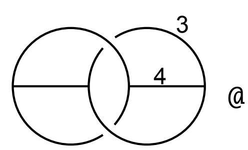

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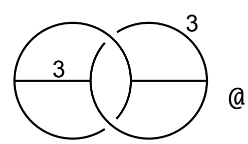

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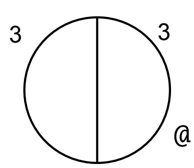

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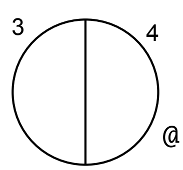

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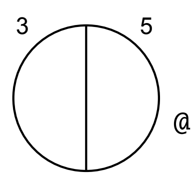

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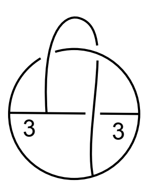

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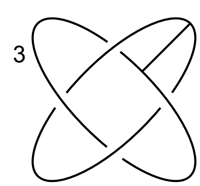

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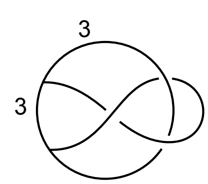

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4 Strategy and outline of finding OE_g

1. Obtain OE_g from allowable 2-suborbifolds in spherical 3-orbifolds.

We already know $OE_g \geq 4(g+1)$ by examples, see [WWZZ], hence to determine OE_g we can assume $|G| > 4(g-1)$.

Definition 4.1. A 2-suborbifold $\mathcal{F}$ in a spherical 3-orbifold S^3/G, with $|G| > 4(g-1)$, is called allowable if its preimage in S^3 is a closed connected surface Σ_g.

Therefore if G acts on (S^3, Σ_g) and realizes OE_g then $\mathcal{F} = \Sigma_g/G \subset S^3/G$ must be an allowable 2-suborbifold. We intend to find extendable actions from allowable 2-suborbifolds in spherical 3-orbifolds and, more weakly, to find the maximum orders of extendable actions from certain information about such allowable 2-suborbifolds.

Suppose we have a spherical 3-orbifold $\mathcal{O}$ and an allowable 2-suborbifold $\mathcal{F} \subset \mathcal{O}$. By Proposition 2.5, $\mathcal{F}$ has underlying space S^2 with four singular points, and moreover $\mathcal{F}$ has a singular type as in the list of Lemma 2.7. Once we know the singular type of $\mathcal{F}$ and the order $\pi_1(\mathcal{O})$, we know the genus of the corresponding closed connected surface $\Sigma_g \subset S^3$ such that $(S^3, \Sigma_g)/G = (\mathcal{O}, \mathcal{F})$ by Lemma 2.8. So if we know the singular types of all allowable 2-orbifolds in $\mathcal{O}$, then we know all Σ_g which admit an extendable action of the group $G \cong \pi_1(\mathcal{O})$; in other words, for a fixed g we know if Σ_g admits an extendable actions of group $G \cong \pi_1(\mathcal{O})$. Hence if we know the singular types of all allowable 2-orbifolds in all spherical 3-orbifolds $\mathcal{O}$, then for a fixed g we know all finite groups $\pi_1(\mathcal{O}) \subset SO(4)$ such that Σ_g admits an extendable action of the group $\pi_1(\mathcal{O})$, and consequently OE_g can be determined.

2. List all allowable 2-suborbifolds in spherical 3-orbifolds.

Definition 4.2. A 2-sphere in a spherical 3-orbifold S^3/G is called candidacy if it intersects the singular graph of S^3/G in exactly four singular points of one of the types listed in Lemma 2.7.

Clearly for each allowable 2-suborbifold $\mathcal{F} \subset S^3/G$, $|\mathcal{F}| \subset S^3/G$ is a candidacy 2-sphere. On the other hand, each candidacy 2-sphere is the underlying space of a non-spherical 2-orbifold $\mathcal{F}$, and we will denote this candidacy 2-sphere by $|\mathcal{F}|$.

We say that a 2-orbifold $\hat{i} : \mathcal{F} \subset S^3/G$ is π_1-surjective if the induced map on the orbifold fundamental groups is surjective.

The process of listing all allowable 2-suborbifolds in spherical 3-orbifolds is divided into two steps:

(i) List all spherical 3-orbifolds containing allowable 2-suborbifolds.

Suppose $\hat{i} : \mathcal{F} \subset S^3/G$ is an allowable 2-suborbifold in a spherical 3-orbifold. Then the preimage of $\mathcal{F}$ must be connected, and by Lemma 2.14 the underlying space of S^3/G is S^3. All spherical 3-orbifolds with underlying space S^3 are listed in Dunbar's lists provided in Section 3. Below we will

denote spherical 3-orbifolds with underline space S^3 by (S^3,Γ), where Γ is the singular set.

Since $\hat{i}:\mathcal{F}\subset(S^3,\Gamma)$ is allowable, the preimage of $\mathcal{F}$ is connected, and by Corollary 2.12, $\hat{i}$ is π_1-surjective. Let (S^3,Γ) be a spherical 3-orbifold with parameters; we will show that, for each 2-suborbifold $\hat{i}:\mathcal{F}\hookrightarrow(S^3,\Gamma)$ such that $|\mathcal{F}|$ is a candidacy 2-sphere and $\hat{i}$ is π_1-surjective, the parameters must satisfy certain equations. Then we can determine the parameters and get a list of spherical 3-orbifold containing allowable 2-suborbifolds which is a small subset of Dunbar's list where the singular sets are relatively simple. Step (i) will be carried in Section 5.

(ii) List all allowable 2-suborbifolds in each spherical 3-orbifold obtained in Step (i).

How to find such 2-suborbifolds? Indeed this is already the question we must face in Step (i). Precisely, this question divides into two subquestions:

(a) How to find candidacy 2-spheres $|\mathcal{F}|$ in a given spherical 3-orbifold (S^3,Γ)?

(b) For each candidacy 2-sphere $|\mathcal{F}|$ we find, how to verify if $\hat{i}$ is π_1-surjective?

A simple and crucial fact in solving Question (a) is provided by Proposition 2.5: For each candidacy 2-sphere $|\mathcal{F}|$, the 2-orbifold $\mathcal{F}$ must bound a handlebody orbifold V; moreover the shape of V is given in Lemma 2.9.

If the singular type is not $(2,2,3,3)$, then V is a regular neighborhood of a singular edge. In this case we can check all the edges to see whether the corresponding singular type is contained in the list of Lemma 2.7.

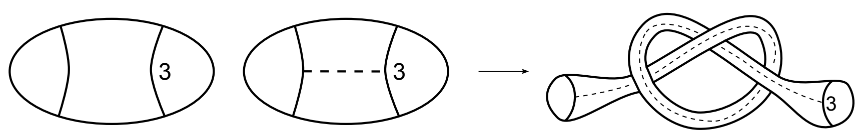


Figure 9

If the singular type is $(2,2,3,3)$, there are two possibilities for the shape of V. The new one can be thought of as a neighborhood of a regular arc with its two ends on singular edges labeled 2 and 3 which will be presented by a dashed arc. If there is a such a dashed arc then we can locally knot this arc in an arbitrary way and obtain infinitely many candidacy 2-spheres, see Figure 9; in this case we only give one such dached arc, and this will be the standard one if it exists (here "standard" means that the boundary of a regular neighborhood of the dashed arc bounds a handlebody orbifold also on the outer side).

To answer Question (b), it is easy to check that all edges and dashed arcs marked as standard (i.e. without any we find are standard, therefore they are automatically allowable (Remark 2.11 (2)).

To verify the π_1-surjectivity of $\hat{i}:\mathcal{F}\subset(S^3,\Gamma)$ for the remaining cases, we are still lucky: all π_1-surjective cases of $\hat{i}:\mathcal{F}\subset(S^3,\Gamma)$ can be verified by the so-called coset enumeration method, and all other cases can be verified by the edge killing method of Lemma 2.13, with three exceptions where Lemma 2.15 will be applied.

5 List of 3-orbifolds containing allowable 2-suborbifolds.

We will establish the equations (and inequalities) which the parameters in Dunbar's list must satisfy in order to contain allowable 2-suborbifolds. We will solve these equations to get all solutions and redrew the pictures of the corresponding 3-orbifolds. Since different solutions often give the same orbifold up to the automorphisms of the orbifold, we will only draw the graphs of non-homeomorphic orbifolds. We will give the detailed computation and verification for the most complicated case in our list Section 7 as an appendix.

Note first all graphs having parameters are contained in Table 6 and Table 7.

Suppose $|\mathcal{F}| \subset |(S^3, \Gamma)|$ is a candidacy 2-sphere. Then $\mathcal{F} \subset (S^3, \Gamma)$ bounds a handlebody orbifold V of given singular type by the discussion in last section.

To determine the parameters, we divide the discussion into two cases:

Case 1. The singular type is not $(2, 2, 3, 3)$; then V is as in Figure 2(a).

Case 2. The singular type is $(2, 2, 3, 3)$, and V is as in Figure 2(a) or as in Figure 2(b).

In Case 1, $\Gamma \cap V$ has two degree 3 vertices, and $\mathcal{F} = \partial V$ has singular type $(2, 2, 2, n), n \geq 3, (2, 2, 3, 4)$ or $(2, 2, 3, 5)$. Hence $\Gamma \cap V$ must be a label 2 arc adding two 'strut segments' with different labels, see Figure 10; (r, s) is either $(2, n)$, or $(3, 4)$, or $(3, 5)$.

Only graphs 15, 19, 20, 21, 22 in Table 7 have more than one strut. So we need only to deal with these five graphs in Case 1.

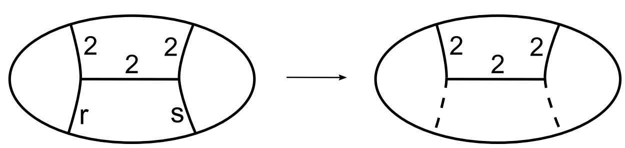


Figure 10

The way to determine the parameters was suggested in the last section:

Suppose $\hat{i} : \mathcal{F} \subset (S^3, \Gamma)$ π_1-surjective. If we kill these two 'strut segments', we obtain $\hat{i} : \mathcal{F}' \subset (S^3, \Gamma')$ which is also π_1-surjective. Since $\pi_1(\mathcal{F}') = \mathbb{Z}_2$, it follows $|\pi_1(S^3, \Gamma')| \leq 2$, therefore Γ' contains no other 'strut' (otherwise $\pi_1((S^3, \Gamma'))$ would not be cyclic), and hence Γ' is a Montesinos link labeled by 2. The double branched cover of S^3 over Γ' must be a 3-manifold N with trivial $\pi_1(N)$ and hence $N = S^3$, that is to say Γ' is a trivial knot by the positive solution of the Smith conjecture. We use the parameters to compute $\pi_1(N)$ and then determine the parameters.

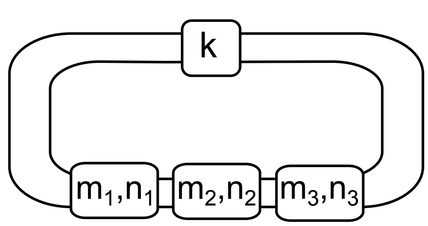


Figure 11

For short we present the graphs 15, 19, 20, 21, 22 by a single graph in Figure 11; the five graphs correspond to the choices $(n, n, 1)$, $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ for (n_1, n_2, n_3) $(n > 1)$; the fact that Γ contains exactly two 'struts' with different labels implies first that at least one m_i is zero (otherwise combining with the assumption $|2m_i| \le n_i$ we will have at most one label on the struts) and then at least one m_i is nonzero (otherwise we have either three struts, or two struts of the same label). Let $(|m_i|, n_i) = d_i$ be the greatest common divisor of $|m_i|$ and n_i, $m_i = m_i' d_i$, $n_i = n_i' d_i$. Then Γ' is the Montesinos link presented by Figure 11, with each (m_i, n_i) replaced by (m_i', n_i'). By a theorem of Montesinos [BZ, Proposition 12.30], the double branched cover of S^3 over Γ' is a Seifert manifold N whose fundamental group has the following presentation:

$$\begin{gathered}\pi_1(N) = \langle x, y, z, t \mid x^{n_1'} t^{m_1'} = y^{n_2'} t^{m_2'} = z^{n_3'} t^{m_3'} = 1, \\ xyzt^{-k} = 1, [x,t] = [y,t] = [z,t] = 1 \rangle\end{gathered}$$

If $m_i = 0$ for some i, then $n_i' = 1$ by definition, and it is easy to see $\pi_1(N)$ is an abelian group. Now $\pi_1(N)$ is trivial if and only if the determinant of the presentation matrix is ± 1. Hence we have

$$kn_1'n_2'n_3' + m_1'n_2'n_3' + n_1'm_2'n_3' + n_1'n_2'm_3' = \pm 1. \qquad (4.1)$$

Dividing $n_1'n_2'n_3'$ on both sides and using the facts that $|2m_i'| \le n_i'$ and $m_i = 0$ for some i, we have

$$|k| \le 1/n_1'n_2'n_3' + |m_1'|/n_1' + |m_2'|/n_2' + |m_3'|/n_3' < 2$$

and then $k = 0, \pm 1$. Combining with the facts that $|2m_i'| \le n_i'$, that Γ has two exactly two 'struts', with different labels, and that $\{d_1, d_2, d_3\} = \{1, 2, d\}, (d > 2)$, or $\{1, 3, 4\}$ or $\{1, 3, 5\}$ by the singular type restrictions, we can find all solutions.

In the following we list all solutions and draw the graphs of the non-homeomorphic orbifolds:

If $(n_1, n_2, n_3) = (2, 3, 3)$, then $(k, m_1, m_2, m_3) = (0, 0, \pm 1, 0)$, $(0, 0, 0, \pm 1)$. We present the picture for the case of $(0, 0, 0, 1)$ (the other cases give homeomorphic orbifolds).

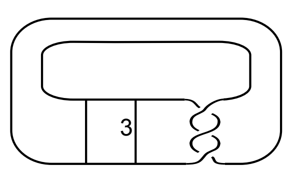


Figure 12

If $(n_1, n_2, n_3) = (2, 3, 4)$ then $(k, m_1, m_2, m_3) = (0, 0, 0, \pm 1)$, $(0, 0, \pm 1, 0)$, $(0, \pm 1, 0, 0)$, $(1, -1, 0, 0)$ or $(-1, 1, 0, 0)$. We present $(0, 0, 0, 1)$, $(0, 0, 1, 0)$ and $(0, 1, 0, 0)$.

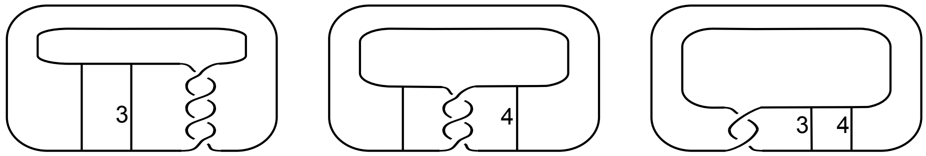


Figure 13

If $(n_1, n_2, n_3) = (2, 3, 5)$ then $(k, m_1, m_2, m_3) = (0, 0, 0, \pm 1)$, $(0, 0, \pm 1, 0)$, $(0, \pm 1, 0, 0)$, $(1, -1, 0, 0)$ or $(-1, 1, 0, 0)$, and we present $(0, 0, 0, 1)$, $(0, 0, 1, 0)$ and $(0, 1, 0, 0)$.

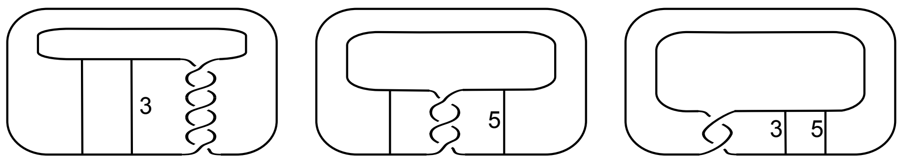


Figure 14

If $(n_1, n_2, n_3) = (2, 2, n)$ then $(k, m_1, m_2, m_3, n) = (0, \pm 1, 0, m'_3 d_3, (1 \mp 2m'_3) d_3)$, $(0, 0, \pm 1, m'_3 d_3, (1 \mp 2m'_3) d_3)$, $(-1, 1, 0, m'_3 d_3, (1 + 2m'_3) d_3)$, $(1, -1, 0, m'_3 d_3, (1 - 2m'_3) d_3)$, $(-1, 0, 1, m'_3 d_3, (1 + 2m'_3) d_3)$ or $(1, 0, -1, m'_3 d_3, (1 - 2m'_3) d_3)$, $n > 1$. We present $(0, 1, 0, m'_3 d_3, (1 - 2m'_3) d_3)$, $m'_3 \neq 0$, and $(0, 1, 0, 0, n)$.

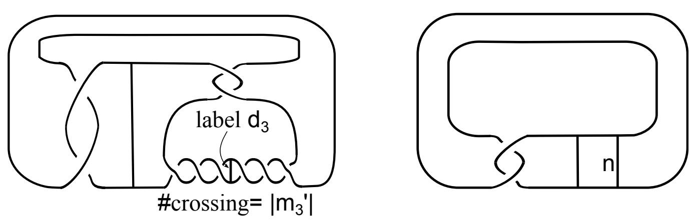


Figure 15

Denote by $\mathcal{O}$ the orbifold presented by the left graph in Figure 15. If we kill the edge which is labeled by d_3, we obtain an orbifold which is fibred over $D^2(2, 2, 1 - 2m'_3)$ with $1 - 2m'_3 \geq 3$ [Du1] (the 2-orbifold with base D^2 and three corner points with labels $(2, 2, 1 - 2m'_3)$. Then after killing the element presenting the fiber, we obtain a surjection $\pi_1(\mathcal{O}) \to \pi_1(D^2(2, 2, 1 - 2m'_3))$. The latter group is non-abelian. But the fundament group of our 2-suborbifold, if it exists, would be $\mathbb{Z}_2 \times \mathbb{Z}_2$ after killing; this means that we cannot find an allowable 2-suborbifold in $\mathcal{O}$.

Finally, if $(n_1, n_2, n_3) = (n, n, 1)$, then $(k, m_1, m_2, n) = (\pm 1, 2m'_1, \mp 1 + 2m'_1, 2 \mp 4m'_1)$, $(\pm 1, \mp 2, 0, 4)$, $(0, 2m'_1, \pm 1 - 2m'_1, 2 \mp 4m'_1)$, $(0, \pm 2, 0, 2n'_1)$, $(0, \pm 4, \mp 3, 12)$, $(0, \pm 6, \mp 5, 15)$, $(\pm 1, \mp 1 + 2m'_2, 2m'_2, 2 \mp 4m'_2)$, $(\pm 1, 0, \mp 2, 4)$, $(0, \pm 1 - 2m'_2, 2m'_2, 2 \mp 4m'_2)$, $(0, 0, \pm 2, 2n'_2)$, $(0, \pm 3, \mp 4, 12)$ or $(0, \pm 5, \mp 6, 15)$, where each $m'_i \neq 0$ and $n'_i \geq 2$, for $i = 1, 2$. We present $(0, -1 - 2m'_2, 2m'_2, 2 + 4m'_2)$, $m'_2 \neq 0$, $(0, 0, 2, 2n'_2)$, $n'_2 \geq 2$, $(0, 4, -3, 12)$ and $(0, 5, -6, 15)$. Note that each of the corresponding graphs isotopes to a simple one as indicated in Figure 16.

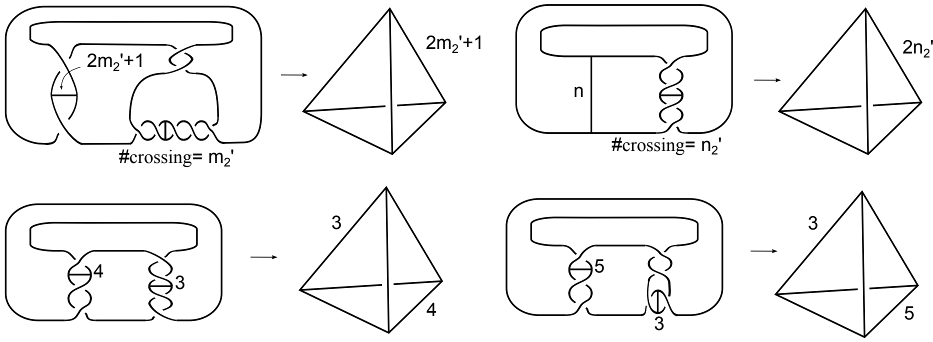


Figure 16

This finishes the discussion for Case 1.

For Case 2, by the same discussion as in Case 1, the graphs we find have the property that if we kill the label 3 'struts' or a label 3 link component, we get a trivial knot labeled by 2.

By this property the link cases (no vertices or struts), including Table 6 now and also the graphs 14 and 16 in Table 7, are easy to handle. Graphs 04, 08, 10, 12, 13 and 14 are ruled out since each of them has only one index. Graphs 03 and 09 are ruled out since after killing an index 3 component (intersecting V), the remaining is not a trivial knot.

In Table 7, there are two further graphs which possibly contain exactly one 'strut', the graphs 17 and 18. Now 17 is ruled out since after killing the possible index 3 'strut' (intersecting V), the remaining is not a trivial knot. Concerning 18, the only possible graph is the link on the upper right hand side of Figure 17 which presents all possible labeled links; the graphs on the lower left hand side come from 02, 05 and 16.

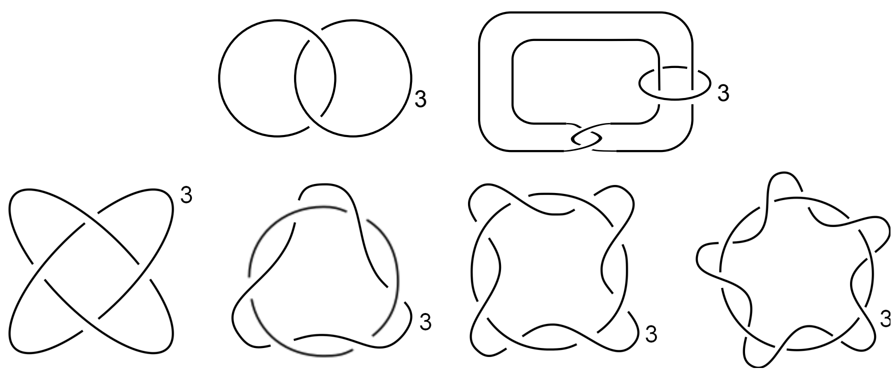


Figure 17

Concerning the possible 'strut' cases, we still have to consider the five graphs discussed in Case 1. We list the solutions and corresponding pictures below as in Case 1. Notice that when $(n_1, n_2, n_3) = (n, n, 1)$, we could have $\{d_1, d_2\} = \{1, 3\}$ or $\{3, 3\}$. Since most of the solutions present the same graph or a mirror image we only picture the graphs of non-homeomorphic orbifolds.

If $(n_1, n_2, n_3) = (2, 3, 3)$ then $(k, m_1, m_2, m_3) = (0, \pm1, 0, 0)$, $(1, -1, 0, 0)$, $(-1, 1, 0, 0)$, $(\pm1, \mp1, 0, \mp1)$, $(0, \pm1, 0, \mp1)$, $(\pm1, \mp1, \mp1, 0)$ or $(0, \pm1, \mp1, 0)$. We present $(0, 1, 0, 0)$ and $(0, -1, 0, 1)$.

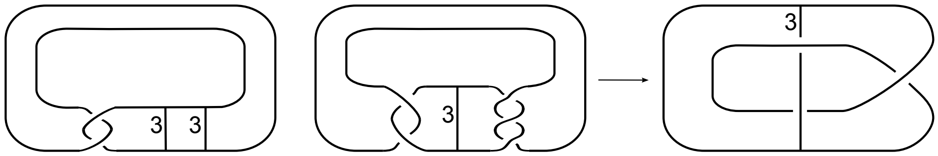


Figure 18

If $(n_1, n_2, n_3) = (2, 3, 4)$, there is no solution.

If $(n_1, n_2, n_3) = (2, 3, 5)$ then $(k, m_1, m_2, m_3) = (\pm 1, \mp 1, 0, \mp 2)$ or $(0, \pm 1, 0, \mp 2)$. We picture $(0, -1, 0, 2)$.

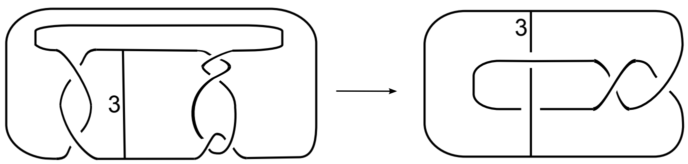


Figure 19

If $(n_1, n_2, n_3) = (2, 2, n)$, there is no solution.

If $(n_1, n_2, n_3) = (n, n, 1)$ then $(k, m_1, m_2, n) = (\pm 1, 0, 0, 3)$, $(0, \pm 1, 0, 3)$ or $(0, 0, \pm 1, 3)$. We picture $(1, 0, 0, 3)$ and $(0, 0, 1, 3)$.

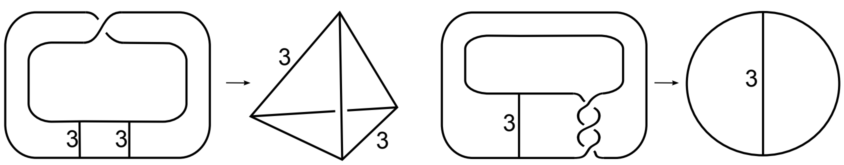


Figure 20

This finishes also the possible 'strut' cases. Concluding, we have found all spherical 3-orbifolds in which an allowable 2-suborbifold might exist.

6 List of allowable 2-suborbifolds

In this section our main result Theorem 6.1 will be presented and proved. We are going to give some explanations and conventions before we state Theorem 6.1.

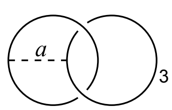
From now on, an edge always means an edge of Γ, the singular set of the orbifold; and a dashed arc is always a regular arc with two ends at two edges of indices 2 and 3.

The primary part of Theorem 6.1 is the list of spherical 3-orbifolds which have survived after the discussion in Section 5. For each 3-orbifold in the list, we first give the order of its fundamental group. Then we use edges and dashed arcs marked by letters a, b, c..., to denote the allowable 2-suborbifolds which are the boundaries of regular neighborhoods of these edges and arcs. Next we write down the singular type of these allowable 2-suborbifold, followed by the corresponding genus which can be computed by Lemma 2.8.

When the singular type is $(2,2,3,3)$, there are two types denoted by I and II, corresponding to Figure 2(a) and Figure 2(b), resp. (so Figure 2(b) gives exactly the dashed arc case, see also Figure 9). If the 2-suborbifold is a knotted one we give a foot notation 'k' to this edge or dashed arc. In the type II cases, if a dashed arc can be chosen as a standard one then also as a knotted one (see Figure 9), and we add a foot notation 'sk' to this arc. We first list the fibred cases of type I and II, then the fibred cases not of type $(2,2,3,3)$, and finally the non-fibred cases.

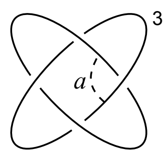
Theorem 6.1. *The following table lists all allowable 2-suborbifolds except those of type II. In the type II case, if there exists an allowable 2-suborbifold associated to a dashed arc we give just one such dashed arc, and this will be standard if there exists a standard one.*

Table 22: Fibred case: type is $(2,2,3,3)$



01

$|G| = 6$
a_{sk}: II, $g = 2$

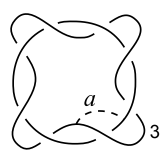


02 or 05 or 16

$|G| = 18$
a_{sk}: II, $g = 4$

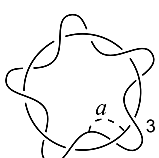
06

$|G| = 48$
a_{sk}: II, $g = 9$



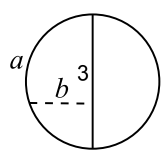
07

$|G| = 144$
a_{sk}: II, $g = 25$



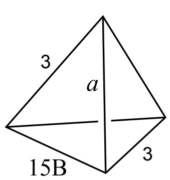
11

$|G| = 720$
a_{sk}: II, $g = 121$



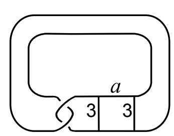
15A

$|G| = 6$
a: I, $g = 2$
b_{sk}: II, $g = 2$



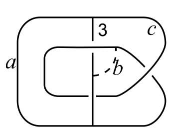
15B

$|G| = 18$
a: I, $g = 4$



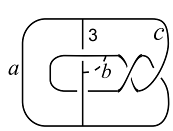
20A

$|G| = 144$
a: I, $g = 25$



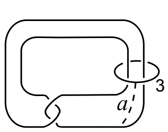
20B

$|G| = 48$
a: I, $g = 9$
b_{sk}: II, $g = 9$
c_k: I, $g = 9$



22A

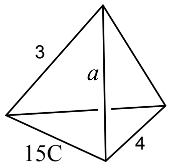
$|G| = 720$
a: I, $g = 121$
b_{sk}: II, $g = 121$
c_k: I, $g = 121$



18

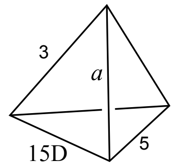
$|G| = 144$
a_{sk}: II, $g = 25$

Table 23: Fibred case: type is not $(2, 2, 3, 3)$



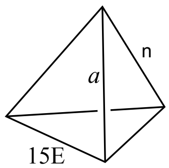
15C

$|G| = 24$
a:(2,2,3,4), $g = 6$



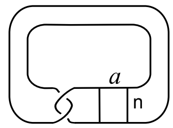
15D

$|G| = 30$
a:(2,2,3,5), $g = 8$



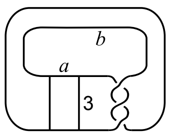
15E

$|G| = 4n$
a:(2,2,2,n)
$g = n - 1$



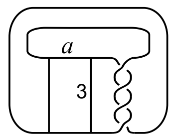
19

$|G| = 4n^2$
a:(2,2,2,n)
$g = (n - 1)^2$



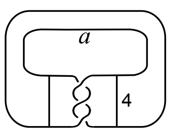
20C

$|G| = 96$
a:(2,2,2,3), $g = 9$
b_k:(2,2,2,3), $g = 9$



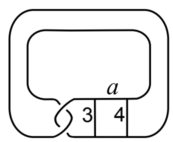
21A

$|G| = 288$
a:(2,2,2,3), $g = 25$



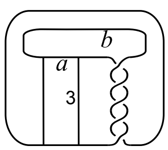
21B

$|G| = 384$
a:(2,2,2,4), $g = 49$



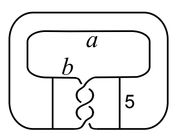
21C

$|G| = 576$
a:(2,2,3,4), $g = 121$



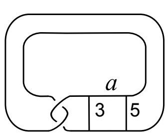
22B

$|G| = 1440$
a:(2,2,2,3), $g = 121$
b_k:(2,2,2,3), $g = 121$



22C

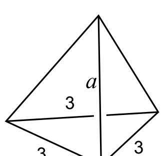
$|G| = 2400$
a:(2,2,2,5), $g = 361$
b_k:(2,2,2,5), $g = 361$



22D

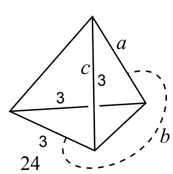
$|G| = 3600$
a:(2,2,3,5), $g = 841$

Table 24: Non-fibred case



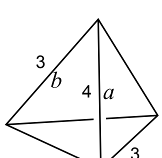
23

$|G| = 96$
a: I, $g = 17$



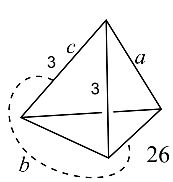
24

$|G| = 60$
a: I, $g = 11$
b_k: II, $g = 11$
c:(2,2,2,3), $g = 6$



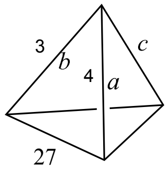
25

$|G| = 576$
a: I, $g = 97$
b:(2,2,2,4), $g = 73$



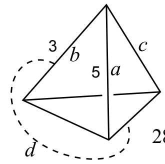
26

$|G| = 24$
a: I, $g = 5$
b_k: II, $g = 5$
c:(2,2,2,3), $g = 3$



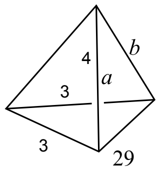
27

$|G| = 48$
a:(2,2,2,3), $g = 5$
b:(2,2,2,4), $g = 7$
c:(2,2,3,4), $g = 11$



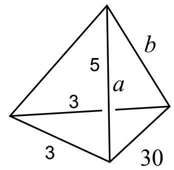
28

$|G| = 120$
a:(2,2,2,3), $g = 11$
b:(2,2,2,5), $g = 19$
c:(2,2,3,5), $g = 29$
d_k: II, $g = 21$



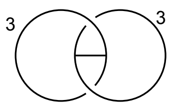
29

$|G| = 192$
a:(2,2,2,3), $g = 17$
b:(2,2,3,4), $g = 41$



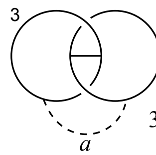
30

$|G| = 7200$
a:(2,2,2,3), $g = 601$
b:(2,2,3,5), $g = 1681$



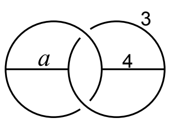
31

$|G| = 288$
No allowable
2-suborbifold



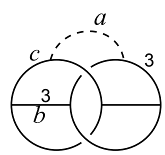
32

$|G| = 24$
a_{sk}: II, $g = 5$



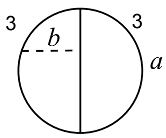
33

$|G| = 1152$
a:(2,2,2,3), $g = 97$



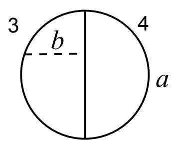
34

$|G| = 120$
a_k: II, $g = 21$
b:(2,2,2,3), $g = 11$
c_k:(2,2,2,3), $g = 11$



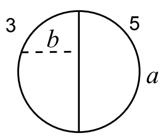
35

$|G| = 12$
a: I, $g = 3$
b_{sk}: II, $g = 3$



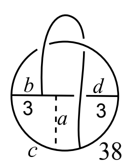
36

$|G| = 24$
a: I, $g = 5$
b_{sk}: II, $g = 5$



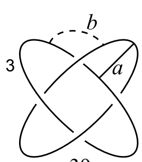
37

$|G| = 60$
a: I, $g = 11$
b_{sk}: II, $g = 11$



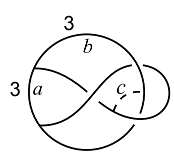
38

$|G| = 2880$
a_k: II, $g = 481$
b, c_k, d_k:(2,2,2,3)
$g = 241$



39

$|G| = 576$
a: I, $g = 97$
b_{sk}: II, $g = 97$



40

$|G| = 1440$
a: I, $g = 241$
b_k: I, $g = 241$
c_{sk}: II, $g = 241$

Proof. How to verify Theorem 6.1?

One can easily check that the list of 3-orbifolds in Theorem 6.1 contains exactly those in Tables 6 and 7 which survived after Section 5, and those in Table 8. By the discussion of Section 5, all allowable 2-orbifolds are contained in one of these 3-orbifolds.

The orders of the fundamental groups can be calculated directly from their group presentations, see [Du2] for the non-fibred case. For the fibered case we used an alternative easier calculation which is as follows. As we discussed in Section 5, for each 3-orbifold (S^3, Γ) in Table 22 and 23 there

is a trivial Montesinos knot $k \subset \Gamma$ of singular index 2. Let $(S^3, \tilde{\Gamma})$ be the double branched cover of (S^3, Γ) over k. In each case $\tilde{\Gamma}$ is a very simple link obtained as the preimage of the struts or of a circle component of Γ, so one can easily calculate $\pi_1(S^3, \tilde{\Gamma})$ and finally obtain $|\pi_1(S^3, \Gamma)| = 2|\pi_1(S^3, \tilde{\Gamma})|$.

To prove Theorem 6.1, we still have to answer the following questions:

I. Why do the marked edges and the dashed arcs give allowable 2-suborbifolds?

II. Why do the non-marked edges and possible candidates of dashed arcs, in orbifolds without marked dashed arcs, do not give allowable 2-suborbifolds?

Concerning I., it is clear by inspection that each marked edge and each dashed arc gives a candidacy 2-sphere $|\mathcal{F}|$ (see Definition 4.1), so it remains to verify that the corresponding 2-suborbifold $\mathcal{F} \subset (S^3, \Gamma)$ is π_1-surjective.

One can easily check that all standard edges and dashed arcs give rise to standard 2-suborbifolds, and then the surjection condition is automatically satisfied. So one has to check only edges and dashed arcs which exist only in a knotted version (i.e. with the subscript 'k').

There are 11 knotted marked edges which are c in 20B, 22A, 34; b in 20C, 22B, 22C, 38, 40; a, e, f in 38; and 5 knotted dashed arcs which are b in 24, 26; d in 28; a in 34; c in 38. The verification of the surjectivity in these 16 cases is based on the so-called coset enumeration method ([Ro], p. 351, Chapter 11) which can be achieved by hand computation for small group orders, and by computer for large orders. We illustrate this in two examples, considering the marked edge c and the dashed arc a in 34:

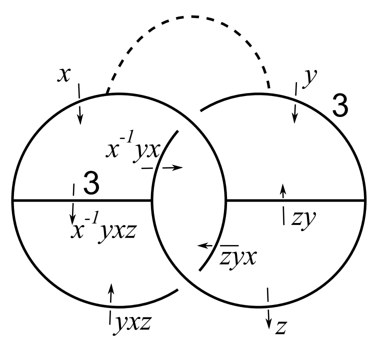


Figure 25

Denote the corresponding 3-orbifold by $\mathcal{O}$. From Figure 25 we obtain the following presentation of the orbifold fundamental group of $\mathcal{O}$:

$$\pi_1(\mathcal{O}) = \langle x, y, z \mid x^2 = y^3 = z^2 = 1, (zy)^2 = (yxz)^2 = (yxzx)^3 = 1 \rangle$$

Denote the 2-suborbifolds corresponding to a and c by $\mathcal{F}_a$ and $\mathcal{F}_c$. Then the image of $\pi_1(\mathcal{F}_a)$ is generated by x and y, and the image of $\pi_1(\mathcal{F}_c)$ is generated by x, zy and yxz. Computing the number of coset of the images of $\pi_1(\mathcal{F}_a)$ and $\pi_1(\mathcal{F}_c)$ in $\pi_1(\mathcal{O})$, we find just one coset and hence the induced homomorphisms are both surjections. This example can be compute by hand since for $\pi_1(\mathcal{F}_a)$ the relation table has just 2 rows, and for $\pi_1(\mathcal{F}_c)$ it has 3 rows.

To answer Question 2, we divide the discussion into two cases:

Case 1. If an edge is not marked, then

(i) the corresponding 2-suborbifold is not a candidacy 2-sphere, or

(ii) the corresponding 2-suborbifold can be mapped to a marked edge by an index preserving automorphism of (S^3, Γ), or

(iii) or the inclusion of $\mathcal{F} \subset (S^3, \Gamma)$ is not π_1-surjective.

Case 2. If there is no dashed arc in a 3-orbifold (S^3, Γ) in the list, then for any dashed arc in (S^3, Γ) giving a candidacy 2-sphere the corresponding 2-suborbifold $\mathcal{F} \subset (S^3, \Gamma)$ is not π_1-surjective.

The verification of Case 1 (i) and (ii) is just by inspection. For example, in 15C only one edge is marked, but the four index 2 edges are transitive under the π-rotation around the axis through the middle points of the edges with indices 3 and 4 and the reflection in the plane through the middle points of these edges. In 33, the candidacy 2-spheres given by the marked edge a and the edge of index 4 are the same up to (orbifold) isotopy.

We will discuss Case 1 (iii) and Case 2. The non-surjectivity of most cases can be derived from the edge killing method stated in Lemma 2.13. We choose 33 as an example:

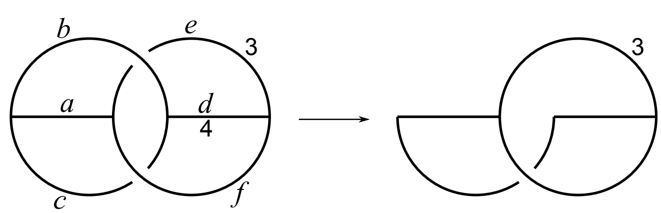


Figure 26

Clearly the edge f does not satisfy the type condition, and up to isotopy a and d present the same 2-suborbifold of type $(2, 2, 2, 3)$. To see that c belongs to Case 1 (iii), kill the edge b; then $\pi_1(\mathcal{F}') = \mathbb{Z}_2$, but $\pi_1((S^3, \Gamma'))$ is a dihedral group of order 6. The same argument shows that also b and e belong to Case 1 (iii).

To see that all possible candidates of dashed arcs belong to Case 2, kill the edge e labeled 3; then $\pi_1(\mathcal{F}') = \mathbb{Z}_2$, but $\pi_1((S^3, \Gamma')) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence we need to mark only the edge a.

There are three orbifolds 23, 29, 30 for which by the edge killing method we do not get the non-π_1-surjectivity of the inclusion corresponding to the dashed arcs. We use Lemma 2.15 to deal with these three cases:

The fundamental groups of these orbifolds are the finite groups $\mathbf{T} \times_{\mathbf{C}_3} \mathbf{T}$, $\mathbf{O} \times_{\mathbf{D}_3} \mathbf{O}$ and $\mathbf{J} \times \mathbf{J}$ of $SO(4)$, see [Du2] for the notations. They can map surjectively to $T \times_{C_3} T$, $O \times_{D_3} O$, $J \times J$ under the 2 to 1 map $SO(4) \to SO(3) \times SO(3)$. For a candidate of a dashed arc, the fundamental group of a regular neighborhood is generated by an order 2 element and an order 3 element. But by Lemma 2.15, any two such elements in the groups $T \times_{C_3} T$, $O \times_{D_3} O$, $J \times J$ cannot generate the whole group. We can also use Lemma 2.15 to deal with other non-fibered orbifolds, for example 23, 25 and 33 whose fundamental groups are given in [Du2] and satisfy the condition of Lemma 2.15. $\square$

Now we state our main results.

Theorem 6.2. *For an extendable finite group action G, if $|G| > 4(g-1)$, all possible relations between $|G|$ and g are listed in the following table. The foot index 'k' means the action is realized only for a knotted embedding, 'sk' means the action can be realized for both a standard and knotted embedding. If the action is realized only for a standard embedding, there is no foot index.*

$\|G\|$	g
$12(g-1)$	$2, 3, 4, 5, 6, 9_{sk}, 11_{sk}, 17, 25, 97, 121_{sk}, 241_{sk}, 601$
$8(g-1)$	$3, 7, 9, 49, 73$
$20(g-1)/3$	$4, 16, 19, 361_{sk}$
$6(g-1)\ I$	$2, 3, 4, 5, 9_{sk}, 11, 17, 25, 97, 121_{sk}, 241_{sk}, 481_k$
$6(g-1)\ II$	$\{2, 3, 4, 5, 9, 11, 25, 97, 121, 241\}_{sk}, 21_k, 481_k$
$24(g-1)/5$	$6, 11, 41, 121$
$30(g-1)/7$	$8, 29, 841, 1681$
$4n(g-1)/(n-2)$	$n-1, (n-1)^2$

Theorem 6.3. *The maximal orders OE_g are given in the following table.*

OE_g	g
$12(g-1)$	$2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241, 601$
$8(g-1)$	$7, 49, 73$
$20(g-1)/3$	$16, 19, 361$
$6(g-1)$	$21, 481$
192	41
7200	1681
$4(\sqrt{g}+1)^2$	$g = k^2, k \neq 3, 5, 7, 11, 19$
$4(g+1)$	*the remaining numbers*

Theorem 6.4. *The maximal orders OE_g^s are given in the following table.*

OE_g^s	g
$12(g-1)$	$2, 3, 4, 5, 6, 9, 11, 17, 25, 97, 121, 241, 601$
$8(g-1)$	$7, 49, 73$
$20(g-1)/3$	$16, 19, 361$
192	41
7200	1681
$4(\sqrt{g}+1)^2$	$g = k^2, k \neq 3, 5, 7, 11, 19$
$4(g+1)$	*the remaining numbers*

Theorem 6.5. *The maximal orders OE_g^k are given in the following table.*

OE_g^k	g
$12(g-1)$	$9, 11, 121, 241$
2400	361
$6(g-1)$	$2, 3, 4, 5, 21, 25, 97, 481$
$4(g-1)$	*the remaining numbers*

Proof. Theorem 6.2 follows from Theorem 6.1; Theorem 6.3 and Theorem 6.4 follow from Theorem 6.2, with some elementary arithmetic.

Note that $4n(g-1)/(n-2)$ will be $12(g-1)$ when $n=3$ and $8(g-1)$ when $n=4$; also, $4n(g-1)/(n-2)$ will be $4(\sqrt{g}+1)^2$ when $g=(n-1)^2$ and $4(g+1)$ when $g=n-1$.

Only the last two rows of the tables in Theorems 6.3 contain infinitely many genera, corresponding to the orbifolds 15E and 19 in Theorem 6.1, or just corresponding to Examples 4.3 and Example 4.4 in [WWZZ].

Theorem 6.5 is derived as below: For every $g>1$, we have an extendable group action with respect to a knotted embedding, of order $4(g-1)$. This can be easily seen from the orbifold 15E in Table 23. We choose a dashed arc connecting a vertex with incident edges labeled $(2,2,2)$ and an edge of index 2. We can knot this arc in an arbitrary way. The boundary of a regular neighborhood of the arc is a knotted 2-suborbifold, and its preimage in S^3 is connected. This gives us an order $4(g-1)$ extendable $(D_{g-1}\oplus\mathbb{Z}_2)$-action on Σ_g.

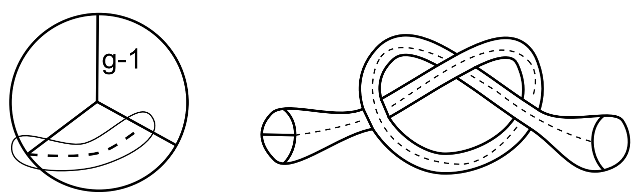


Figure 27

By Theorem 6.2, we know all cases with $|G|>4(g-1)$, we reach Theorem 6.5. $\square$

We define two actions of a finite group G to be equivalent if the corresponding groups of homeomorphisms of (S^3,Σ_g) are conjugate (i.e., allowing isomorphisms of G).

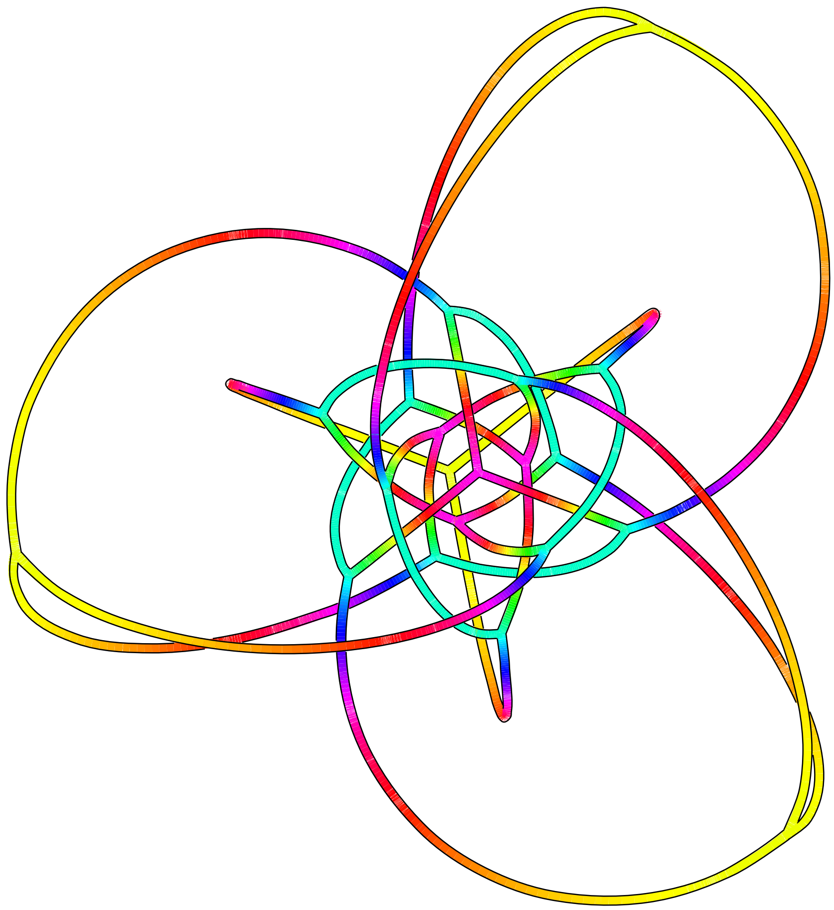
By the proof of Theorem 6.1 and the tables above, we have

Theorem 6.6. *There are only finitely many types of actions of G on (S^3,Σ_g) such that $|G|>4(g-1)$ and the handlebody orbifold bounded by Σ_g/G is not of type II. In particular there are only finitely many types of actions of G on (S^3,Σ_g) realizing OE_g for $g\neq 21,481$.*

Example 6.7. Picture 28 shows a knotted handlebody of genus $g=11$ which is invariant under a group action of order 120 of S^3, corresponding to edge c in 34 (Table 23). All points are colored by their distance from the origin.

The group is isomorphic to $A_5\times\mathbb{Z}_2$, where we consider the alternating group A_5 as the orientation preserving symmetry group of the 4-dimensional regular Euclidean simplex, and $\mathbb{Z}_2$ is generated by $-\mathrm{id}$ on R^4. Let the 4-simplex be centered at the origin of E^4 and inscribed in the unit sphere S^3. The radial projection of its boundary to S^3 gives a tesselation of S^3 by 5 tetrahedra invariant under the action of A_5. We present one of these tetrahedron in Figure 29.

Imagine the figure has spherical geometry. O is the center of the tetrahedron, F is the center of triangle $\triangle BCD$, E is the middle points of BC, M is the middle points of BO, N is the middle points of EF. The orbit of the geodesic MN under the group action of A_5 joins to a graph in S^3; note that this graph is invariant also under $-\mathrm{id}$ on S^3. Projecting to E^3, Picture 28 shows this graph and the boundary surface of the regular neighborhood of the projected image.



Picture 28

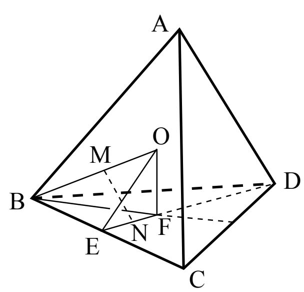


Figure 29

7 Appendix: A detailed verification

As promised in the beginning Section 5, for the most laborious case, $(n, n, 1)$ in Case 1, we will compute all solutions of the equations and verify that the

graphs we drew present all homeomorphism types given by these solutions. The treatments of all other cases are similar but much simpler.

For the case $(n, n, 1)$ we have $n_3 = 1$, $m_3 = 0$, $d_3 = 1$, and the equation (4.1) becomes

$$kn'_1n'_2 + n'_1m'_2 + n'_2m'_1 = \pm 1 \qquad (4.2).$$

We will substitute $k = 0, \pm 1$ in (4.2) to get the solutions. We will repeatedly use the following facts

$$(n'_1, n'_2) = 1 \ (*), \ \ 2|m_i| = 2d_i|m'_i| \le d_in'_i = n > 2, \ \ i = 1, 2 \ (**) \qquad (4.3)$$

We assume first $d_1 < d_2$ and divide the discussion into two cases:

Case (1) $d_1 = 2$, $d_2 > 2$:

Note first that $m'_1 \neq 0$, otherwise $n'_1(kn'_2+m'_2) = \pm 1$ implies that $n'_1 = 1$, so $n = 2$ which is impossible.

(i) When $k = 1$, since $n'_1n'_2 + n'_1m'_2 + n'_2m'_1 \ge 0$, we must have $n'_1n'_2 + n'_1m'_2 + n'_2m'_1 = 1$ by (4.2). Then

$$1 = n'_2m'_1 + n'_1(n'_2 + m'_2) \ge n'_2m'_1 + 2|m'_1|(n'_2 + m'_2) \ge |m'_1|(n'_2 + 2m'_2) \ge 0.$$

If $n'_2 + 2m'_2 = 0$, by the equation we have $(-m'_2)(n'_1 + 2m'_1) = 1$, hence $n'_1 = 1 - 2m'_1$, $m'_2 = -1$, $n'_2 = 2$. We have the solution $(k, m_1, m_2, n) = (1, 2m'_1, -1 + 2m'_1, 2 - 4m'_1)$, $m'_1 \neq 0$.

If $n'_2+2m'_2 > 0$, then $n'_2+2m'_2 = 1$. The inequalities should be equalities, and $m'_1 = -1$, $n'_1 = 2$. We get the solution $(k, m_1, m_2, n) = (1, -2, 0, 4)$.

(ii) When $k = -1$, since $-n'_1n'_2 + n'_1m'_2 + n'_2m'_1 \le 0$ by (4.3)(**), we must have $n'_1n'_2 - n'_1m'_2 - n'_2m'_1 = 1$ by (4.2). Then

$$1 = -n'_2m'_1+n'_1(n'_2-m'_2) \ge -n'_2m'_1+2|m'_1|(n'_2-m'_2) \ge |m'_1|(n'_2-2m'_2) \ge 0.$$

If $n'_2 - 2m'_2 = 0$, by the equation we have $m'_2(n'_1 - 2m'_1) = 1$, hence $n'_1 = 1 + 2m'_1$, $m'_2 = 1$, $n'_2 = 2$. We have the solution $(k, m_1, m_2, n) = (-1, 2m'_1, 1 + 2m'_1, 2 + 4m'_1)$, $m'_1 \neq 0$.

If $n'_2-2m'_2 > 0$, then $n'_2-2m'_2 = 1$. The inequalities should be equalities, we have $m'_1 = 1$, $n'_1 = 2$. We have $(k, m_1, m_2, n) = (-1, 2, 0, 4)$.

(iii) When $k = 0$ then $n'_1m'_2 + n'_2m'_1 = \pm 1$. By (4.3)(**) we have $n'_2(d_2m'_2/2 + m'_1) = \pm 1$ which implies that $n'_2 = 1$ or 2.

If $n'_2 = 1$, then $d_2 = n$, and $|m_2| \le d_2/2$. Hence $(m_2, d_2) = d_2$ implies that $m_2 = 0$, $m'_1 = \pm 1$. We have the solution $(k, m_1, m_2, n) = (0, \pm 2, 0, 2n'_1)$, $n'_1 \ge 2$.

If $n'_2 = 2$ then $2d_2 = n$ and $(m_2, 2d_2) = d_2$ implies $m'_2 = \pm 1$ and $|m_2| = d_2 = n'_1$. Plugging $m'_2 = 1$ into $n'_1m'_2 + 2'm'_1 = \pm 1$, we have $n'_1 + 2'm'_1 = \pm 1$, which implies that the right side must be 1, and we have $n'_1 = 1 - 2'm'_1$. Similarly plugging $m'_2 = -1$ into $n'_1m'_2 + 2'm'_1 = \pm 1$ we get the solution $-n'_1 = -1 - 2'm'_1$. So have have two solutions $n'_1 = 1 \mp 2'm'_1$, and finally $(k, m_1, m_2, n) = (0, 2m'_1, \pm 1 - 2m'_1, 2 \mp 4m'_1)$, $m'_1 \neq 0$.

Case (2) $d_1 = 3$, $d_2 > 3$.

Suppose $(d_1, d_2) = (3, 4)$. By $g.c.d(n'_1, n'_2) = 1$ and $n = d_1 n'_1 = d_2 n'_2$, we have $n'_2 = 3$, $n'_1 = 4$ and hence $n = 12$. Plug $n'_2 = 3$ and $n'_1 = 4$ into (4.2) to get the solution $k = 0, m_1 = \pm 4, m_2 = \mp 3$ by applying (4.3). So we have the solutions $(k, m_1, m_2, n) = (0, \pm 4, \mp 3, 12)$.

If $(d_1, d_2) = (3, 5)$, by aa similar argument we have $(k, m_1, m_2, n) = (0, \pm 6, \mp 5, 15)$.

Assuming then $d_1 > d_2$, symmetrically we get $(\pm 1, \mp 1 + 2m'_2, 2m'_2, 2 \mp 4m'_2)$, $(\pm 1, 0, \mp 2, 4)$, $(0, \pm 1 - 2m'_2, 2m'_2, 2 \mp 4m'_2)$, $(0, 0, \pm 2, 2n'_2)$, $(0, \pm 3, \mp 4, 12)$ and $(0, \pm 5, \mp 6, 15)$.

Figure 21 shows how a solution can be transformed to another one by an edge move or a mirror image, hence the graphs given by $(k, m_1, m_2, n) = (\pm 1, 2m'_1, \mp 1 + 2m'_1, 2 \mp 4m'_1)$ and $(0, 2m'_1, \pm 1 - 2m'_1, 2 \mp 4m'_1)$ present the same type of 3-orbifold.

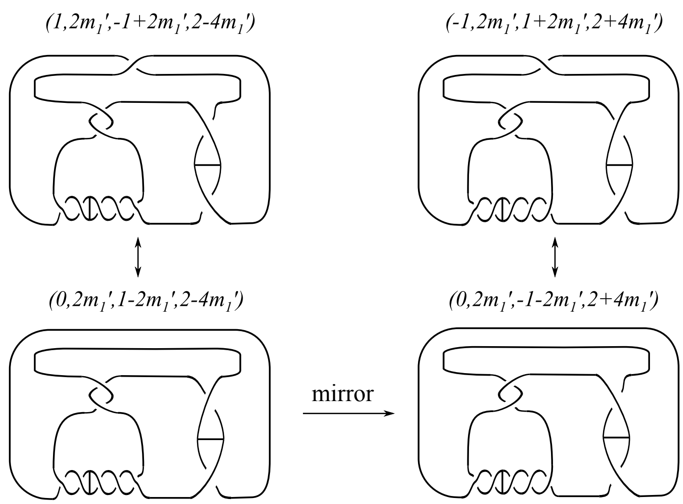


Figure 30

Similarly the solutions $(\pm 1, \mp 1 + 2m'_2, 2m'_2, 2 \mp 4m'_2)$ and $(0, \pm 1 - 2m'_2, 2m'_2, 2 \mp 4m'_2)$ give the same type of orbifold presented by the first graph in Figure 16 which is clearly equivalent to the left-lower graph in Figure 21, hence these eight solutions given the same orbifold presented by the first graph in Figure 16.

By the same argument, the solutions $(0, \pm 4, \mp 3, 12)$ and $(0, \pm 3, \mp 4, 12)$ give the same orbifold presented by the left-lower graph in Figure 16, and the solutions $(0, \pm 5, \mp 3, 15)$ and $(0, \pm 3, \mp 5, 15)$ give the same orbifold presented by the right-lower graph in Figure 16.

Finally by the transformation given on the left side of Figure 30, we see that the orbifolds given by the solutions $(\pm 1, \mp 2, 0, 4)$ and $(\pm 1, 0, \mp 2, 4)$ are special cases of those given by $(0, \pm 2, 0, 2n'_1)$ and $(0, 0, \pm 2, 2n'_2)$, therefore all these eight solutions are presented by the orbifold presented by the upper-right graph in Figure 16.

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